On a Bernstein-Like Trigonometric Basis without Shape Parameters for Cubic Bézier Curves

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ABSTRACT:

In this paper, we introduce a new trigonometric blending basis of Bernstein type without shape parameters for the construction of Bézier-like curves. Fundamental properties of the new basis are rigorously established, including non-negativity, partition of unity, and endpoint interpolation.

A theoretical comparison with the classical Bernstein basis is provided, highlighting similarities in geometric behavior and differences in approximation capability. We used to assess practical performance, least-squares curve fitting and several numerical examples are presented. These results indicate that the proposed parameter-free trigonometric basis offers an effective alternative to classical Bernstein polynomials, combining strong geometric properties with improved approximation quality for curve modeling applications.

Keywords:

Trigonometric blending basis, Bernstein-like basis, Cubic Bezier curve, Least-squares method, Curve fitting.

تاريخ إيداع البحث: 2026/02/02

تاريخ قبوله للنشر: 2026/02/25

قاعدة مثلثية شبيهة بيرنشتاين دون مَعلَمة الشكل لمنحنيات بيزييه التكعيبية

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الملخص:

يهدف البحث إلى اقتراح قاعدة مزج مثلثية جديدة من نمط بيرنشتاين، خالية من وسيط الشكل، لاستخدامها في بناء منحنيات شبيهة بمنحنيات بيزييه.

جرى إثبات الخصائص الأساسية لهذه القاعدة الجديدة بدقة رياضية، بما في ذلك عدم السلبية، وخاصية تجزئة الوحدة، واستيفاء النهايتين. كما جرت المقارنة النظرية مع قاعدة بيرنشتاين التقليدية، لتسليط الضوء على أوجه التشابه في السلوك الهندسي والاختلافات في القدرة التقريبية.

استخدمنا لتقييم الأداء العملي طريقة التقريب بالمربعات الصغرى لملاءمة المنحنيات، وجرى عرض العديد من الأمثلة العددية.

تشير النتائج إلى أن القاعدة المثلثية المقترحة الخالية من معلمات الشكل تمثّل بديلًا فعّالًا لكثيرات حدود بيرنشتاين التقليدية، إذ تجمع بين الخصائص الهندسية القوية وتحسين جودة التقريب في تطبيقات نمذجة المنحنيات.

الكلمات المفتاحية:

قاعدة المزج المثلثية، قاعدة شبيهة بيرنشتاين، منحنى بيزييه التكعيبي، طريقة المربعات الصغرى، ملاءمة المنحنيات.

1- Introduction

Blending functions play a fundamental role in the mathematical representation of curves and surfaces in computer-aided geometric design (CAGD). By assigning smoothly varying weights to control points, blending functions enable the construction of parametric curves that are both geometrically intuitive and numerically stable. A well-designed blending basis ensures essential properties such as non-negativity, partition of unity, and affine invariance, all of which are crucial for predictable shape manipulation [1,2].

Among the various blending bases, the Bernstein basis occupies a central position. It forms the foundation of Bezier curves, particularly cubic Bézier curves, which are extensively used in computer graphics, CAD/CAM systems, font design, and animation. The popularity of the Bernstein basis stems from its strong shape-preserving properties, including the convex hull and variation-diminishing properties, as well as its direct geometric interpretation through control points [1,3]. These characteristics make cubic Bezier curves a standard tool for interactive curve modeling.

Despite these advantages, classical polynomial Bernstein bases may exhibit limitations when approximating data with pronounced curvature, oscillatory behavior, or complex geometric features. In such cases, polynomial representations often require higher degrees to achieve the desired accuracy, which can adversely affect computational efficiency and numerical stability [4].

This observation has motivated the exploration of generalized and alternative blending bases that preserve desirable geometric properties while offering enhanced flexibility and approximation performance.

Recent advancements have introduced generalized blended trigonometric Bernstein basis functions, retaining classical geometric properties while enabling additional control over curve shape and continuity conditions [5-7]. These generalized trigonometric bases have been shown to facilitate improved geometric flexibility and can represent curves more closely

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matching control polygons compared to traditional polynomial bases, especially for shapes with inherent trigonometric characteristics Investigations into quintic and higher-order trigonometric curve formulations further demonstrate the applicability of trigonometric bases in modern geometric modeling. [8-9]
In this paper, we investigate the use of a trigonometric blending basis as an alternative to the standard Bernstein basis for cubic Bezier curve construction. The proposed trigonometric basis retains key blending properties such as non-negativity and partition of unity.
By replacing the classical polynomial Bernstein basis with a trigonometric counterpart, we aim to enhance approximation quality, particularly in the least-squares sense, without increasing the number of control parameters.
Numerical experiments and comparative analysis demonstrate that the trigonometric basis can provide improved approximation performance over the classical Bernstein basis for certain classes of data, while maintaining geometric interpretability and design flexibility.

2- Bezier curves

Bezier curves is a parametric curve $\mathrm{P}(t)$ that is a polynomial function of the parameter t. The degree of the polynomial depends on the number of points used to define the curves. The control polygon of the Bezier curve is the polygon obtained when the control points are connected, in their natural order, with straight segments [10-12].

Definition 1

A Bezier curve of degree n is defined by control points $\mathrm{P}_0, \mathrm{P}_1, \dots, \mathrm{P}_n$ as follows: [1]

$$\mathrm{P}(t) = \sum_{i=0}^{n} \mathrm{P}_i(t) B_{n,i}(t)\,,\ 0 \le t \le 1$$

where $B_{n,i}(t)$ are Bernstein basis polynomials;

$$B_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i}.$$

The Bezier curve can be represented in a very compact and elegant way as

$$P(t) = (1 - t + tE)^n P_0,$$

where E is the shift operator defined by $EP_i = P_{i+1}$.The definition of E implies

$$EP_0 = P_1$$

and

$$E^i P_0 = P_i$$

and the Bezier curve can now be written as follows:

$$\begin{aligned} P(t) &= \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} P_i \\ &= \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} E^i P_0 \\ &= \sum_{i=0}^{n} \binom{n}{i} (tE)^i (1-t)^{n-i} P_0 \\ &= (tE + (1-t))^n P_0. \end{aligned}$$

For $n = 3$, we get [9, 10]

$$\begin{aligned} P(t) &= (1 - t + tE)^3 P_0 \\ &= ((1-t)^3 + 3(1-t)^2 tE + 3(1-t)t^2E^2 + t^3E^3)P_0 \\ &= (1-t)^3 P_0 + 3(1-t)^2 tEP_0 + 3(1-t)t^2E^2P_0 + t^3E^3P_0 \\ &= (1-t)^3 P_0 + 3(1-t)^2 tP_1 + 3(1-t)t^2P_2 + t^3P_3 . \end{aligned}$$

3− Properties of Bernstein polynomials [13, 14]

a) **Non−negativity:**

For $0 \le t \le 1$, $B_i^n(t)$ are all non−negative.

b) **Partition of unity:**

$$\sum_{i=0}^{n} B_i^n(t) = 1; \ 0 \le t \le 1 .$$

c) **Symmetry:**

$$B_i^n(t) = B_{n-i}^n(t)(1-t).$$

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d) **Maximum location:**

$B_i^n(t)$ attains its maximum at $t = \frac{i}{n}$.

e) **Recursion formula:**

$B_i^n(t) = (1-t)B_i^{n-1}(t) + tB_{i-1}^{n-1}(t)$

4− A Bernstein−Like trigonometric Basis

In computational geometry, computer graphics, and computer−aided design, Bezier curves and their generalizations remain fundamental tools for modeling smooth curves and surfaces [15,16].

These curves are defined through basis functions that determine how control points influence the resulting geometry. Among the various choices, parameter−free bases—most notably the Bernstein−type bases used in classical Bézier curves—continue to be widely adopted due to their simplicity, numerical stability, and strong geometric properties [16,17]. Such bases depend only on the curve parameter $t \in [0,1]$ and the polynomial degree, without additional shape−controlling parameters. This absence of shape parameters leads to predictable and intuitive geometric behavior. In particular, parameter−free bases satisfy the convex hull property, ensuring that the curve remains within the control polygon [17]. They are also affine invariant, meaning that geometric transformations of the control points are directly reflected in the curve [16]. Moreover, efficient algorithms such as de Casteljau−type subdivision schemes support stable evaluation, refinement, and degree elevation [18].

Owing to these advantages, parameter-free bases remain standard in modern applications including CAD, animation, font design, and robotics [15-19].

Definition 2

We define our proposed cubic basis as follows:

$$B_0^3(t) = \cos^2\left(\frac{\pi t}{2}\right)\left(1 - \sin\left(\frac{\pi t}{2}\right)\right)$$
$$B_1^3(t) = \sin\left(\frac{\pi t}{2}\right)\cos^2\left(\frac{\pi t}{2}\right)$$
$$B_2^3(t) = \cos\left(\frac{\pi t}{2}\right)\sin^2\left(\frac{\pi t}{2}\right)$$
$$B_3^3(t) = \sin^2\left(\frac{\pi t}{2}\right)\left(1 - \cos\left(\frac{\pi t}{2}\right)\right).$$

Figure 1 shows the graph of the basis above.

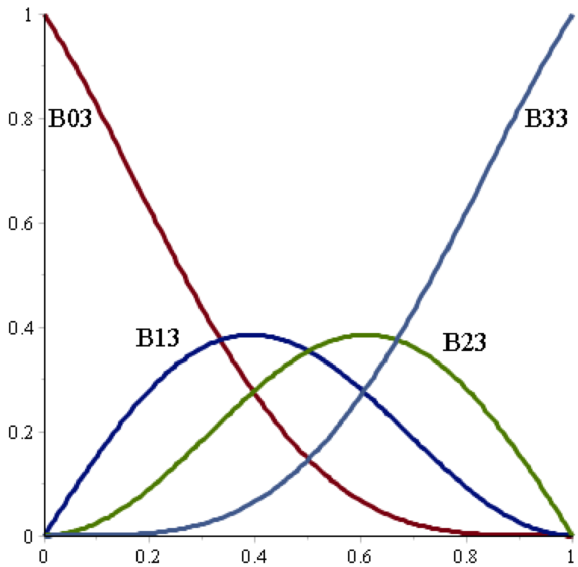

Figure 1

Trigonometric blending bases of Bernstein type have been widely investigated as effective alternatives to classical polynomial bases in computer-aided geometric design, due to their ability to better represent curved and arc-like shapes while preserving essential geometric properties

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such as non-negativity, partition of unity, and endpoint interpolation [20,21].
Similar constructions and motivations can be found in recent studies on generalized and trigonometric Bezier-type bases [22,23].
The proposed trigonometric basis has the following properties:

a) Partition of unity:

$$\sum_{i=0}^{3} B_i^3(t) = 1,\ t \in [0,1]$$

Proof:

$$\begin{aligned}\sum_{i=0}^{3} B_0^3 &= \cos^2\left(\frac{\pi t}{2}\right)\left(1-\sin\left(\frac{\pi t}{2}\right)\right)+\sin\left(\frac{\pi t}{2}\right)\cos^2\left(\frac{\pi t}{2}\right)\\ &\quad+\cos\left(\frac{\pi t}{2}\right)\sin^2\left(\frac{\pi t}{2}\right)+\sin^2\left(\frac{\pi t}{2}\right)\left(1-\cos\left(\frac{\pi t}{2}\right)\right)\\ &= \cos^2\left(\frac{\pi t}{2}\right)-\cos^2\left(\frac{\pi t}{2}\right)\sin\left(\frac{\pi t}{2}\right)+\sin\left(\frac{\pi t}{2}\right)\cos^2\left(\frac{\pi t}{2}\right)\\ &\quad+\cos\left(\frac{\pi t}{2}\right)\sin^2\left(\frac{\pi t}{2}\right)+\sin^2\left(\frac{\pi t}{2}\right)-\sin^2\left(\frac{\pi t}{2}\right)\cos\left(\frac{\pi t}{2}\right)\\ &= \cos^2\left(\frac{\pi t}{2}\right)+\sin^2\left(\frac{\pi t}{2}\right)=1.\end{aligned}$$

b) Symmetry:

$$B_i^3(t) = B_{3-i}^3(1-t),\ i = 0, 1,\ 2,\ 3.$$

Proof:

Using the co−function identities:

$$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$$
$$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$$

for all $\theta \in \mathbb{R}$, we find that

$$\begin{aligned}B_0^3(t) &= \cos^2\left(\frac{\pi t}{2}\right)\left(1-\sin\left(\frac{\pi t}{2}\right)\right)\\ &= \sin^2\left(\frac{\pi}{2}-\frac{\pi t}{2}\right)\left(1-\cos\left(\frac{\pi}{2}-\frac{\pi t}{2}\right)\right)\end{aligned}$$

$$= \sin^2\left(\frac{\pi(1-t)}{2}\right)\left(1-\cos\left(\frac{\pi(1-t)}{2}\right)\right)$$
$$= B_3^3(1-t).$$
$$B_1^3(t) = \sin\left(\frac{\pi t}{2}\right)\cos^2\left(\frac{\pi t}{2}\right)$$
$$= \cos\left(\frac{\pi}{2}-\frac{\pi t}{2}\right)\sin^2\left(\frac{\pi}{2}-\frac{\pi t}{2}\right)$$
$$= \cos\left(\frac{\pi(1-t)}{2}\right)\sin^2\left(\frac{\pi(1-t)}{2}\right) = B_2^3(1-t).$$
$$B_2^3(t) = \cos\left(\frac{\pi t}{2}\right)\sin^2\left(\frac{\pi t}{2}\right)$$
$$= \sin\left(\frac{\pi}{2}-\frac{\pi t}{2}\right)\cos^2\left(\frac{\pi}{2}-\frac{\pi t}{2}\right)$$
$$= \sin\left(\frac{\pi(1-t)}{2}\right)\cos^2\left(\frac{\pi(1-t)}{2}\right) = B_1^3(1-t).$$
$$B_3^3(t) = \sin^2\left(\frac{\pi t}{2}\right)\left(1-\cos\left(\frac{\pi t}{2}\right)\right)$$
$$= \cos^2\left(\frac{\pi}{2}-\frac{\pi t}{2}\right)\left(1-\sin\left(\frac{\pi}{2}-\frac{\pi t}{2}\right)\right)$$
$$= \cos^2\left(\frac{\pi(1-t)}{2}\right)\left(1-\sin\left(\frac{\pi(1-t)}{2}\right)\right) = B_0^3(1-t).$$

c) Non−negativity:

$$B_i^3(t) \geq 0;\ i = 0,1,2,3;\ \ \forall\, t \in [0,1]$$

Proof:

If $t \in [0,1]$, then we have

$\sin\left(\frac{\pi t}{2}\right) \geq 0$, $\cos\left(\frac{\pi t}{2}\right) \geq 0$, $1-\sin\left(\frac{\pi t}{2}\right) \geq 0$ and $1-\cos\left(\frac{\pi t}{2}\right) \geq 0$. Since each $B_i^3(t)$ is expressed as a product of non−negative factors, it follows that $B_i^3(t) \geq 0$ for all $t \in [0,1]$.

d) Endpoint interpolation:

The formula of cubic Bezier curve using the proposed trigonometric basis is

$$\mathrm{P}(t) = B_0^3(t)\mathrm{P}_0 + B_1^3(t)\mathrm{P}_1 + B_2^3(t)\mathrm{P}_2 + B_3^3(t)\mathrm{P}_3.$$

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Where P_0, P_1, P_2, P_3 are control points.

We have

$$P(0) = P_0 \text{ and } P(1) = P_3.$$

Proof:

$$P(0) = B_0^3(0)P_0 + B_1^3(0)P_1 + B_2^3(0)P_2 + B_3^3(0)P_3$$
$$= 1P_0 + 0P_1 + 0P_2 + 0P_3 = P_0.$$

And

$$P(1) = B_0^3(1)P_0 + B_1^3(1)P_1 + B_2^3(1)P_2 + B_3^3(1)P_3$$
$$= 0P_0 + 0P_1 + 0P_2 + 1P_3 = P_3.$$

e) Endpoint tangency:

We have

$$P'(0) = \frac{\pi}{2}(P_1 - P_0)$$
$$P'(1) = \frac{\pi}{2}(P_3 - P_2)$$

Proof:

Computing the derivatives of basis at $t = 0$, we get:

$$\frac{d}{dt}B_0^3(t=0) = -\frac{\pi}{2}$$
$$\frac{d}{dt}B_1^3(t=0) = \frac{\pi}{2}$$
$$\frac{d}{dt}B_2^3(t=0) = 0$$
$$\frac{d}{dt}B_3^3(t=0) = 0\,.$$

Hence

$$P'(0) = B'^3_0(0)P_0 + B'^3_1(0)P_1 = -\frac{\pi}{2}P_0 + \frac{\pi}{2}P_1 = \frac{\pi}{2}(P_1 - P_0)$$

Similarly, the derivatives of basis at $t = 1$ are:

$$\frac{d}{dt}B_0^3(t=1) = 0$$
$$\frac{d}{dt}B_1^3(t=1) = 0$$

$$\frac{d}{dt}B_2^3(t=1) = -\frac{\pi}{2}$$
$$\frac{d}{dt}B_3^3(t=1) = \frac{\pi}{2}.$$

So, we have

$P'(1) = B'^3_2(1)P_2 + B'^3_3(1)P_3 = -\frac{\pi}{2}P_2 + \frac{\pi}{2}P_3 = \frac{\pi}{2}(P_3 - P_2).$

This means that the cubic Bezier curve with proposed trigonometric basis has two endpoint tangents: the line segment P_0P_1 is tangent line to the curve $P(t)$ at $t = 0$, while the line segment P_3P_2 is tangent line to the curve $P(t)$ at $t = 1$.

f) Smoothness:

$$B_i^3(t) \in C^\infty[0,1]; i = 0,1,2,3.$$

Proof:

Each $B_i^3(t)$ for $t \in [0,1]$ and $i = 0,1,2,3$ is a finite product and sum of sine and cosine functions, which are analytic. So, the property holds.

5- Curve fitting:

Curve fitting is the process of constructing a curve from a given family of functions that best approximates a set of data points, according to a specified criterion, usually by minimizing an error measure such as the least-squares error [24, 25].

Curve fitting in the sense of least square method using cubic Bezier curve is given as follows [25,26]:

Let $Q_i = \{x_i, y_i\}_{i=1}^n$ be a set of data points, then our goal is to determine control points $P_0, P_1, \dots, P_n$ so that the Bezier curve

$$P(t) = \sum_{i=0}^n B_i^n(t)P_i \,; t \in [0,1],$$

best approximates the data, where $B_i^n(t)$ are Bernstein basis polynomials of degree n [25]. The problem is formulated as follows:

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$$Lsq(\mathrm{P}_1, \mathrm{P}_2) = \sum_{i=0}^{n} \|\mathrm{P}(t_i) - Q_i\|^2 \rightarrow \min \tag{1}$$

where P_0 and P_3 are fixed. The problem (1) reduces to the solution of the following linear system [25−27]:

$$\frac{\partial}{\partial \mathrm{P}_i} Lsq(\mathrm{P}_1, \mathrm{P}_2) = 0; \quad i = 1, 2. \tag{2}$$

After establishing the optimality conditions in equation (2), the next step is to explicitly assemble the resulting linear system by substituting the Bezier curve expression into the least-squares functional. This leads to a system of linear equations in terms of the unknown control points P_1 and P_2, where the coefficients depend on the Bernstein basis functions evaluated at the parameter values t_i [28, 29].

The parameter values t_i are typically computed using a suitable parameterization method, such as chord-length or centripetal parameterization, to ensure numerical stability and an accurate approximation [30, 31]. Once the linear system is constructed, it can be solved using standard numerical techniques, such as Gaussian elimination or matrix factorization methods.

The obtained solutions for P_1 and P_2, together with the fixed endpoints P_0 and P_3, fully determine the cubic Bezier curve that best fits the given data in the least-squares sense [29-32].

Finally, the quality of the fitted curve can be evaluated by computing the residual error or by visually comparing the curve with the original data points, allowing for further refinement if necessary [31-32**]**.

Example 1

Use cubic Bezier curve to fit the set of data points

$$\mathrm{S} = \{(0, 0), (1, 2), (2, 3), (3, 2.5), (4, 1)\}$$

Solution:

First, we will solve the problem using cubic Bezier curve with Bernstein basis. The first and the last control points are fixed, so we have

$P_0 = (0,0)$ and $P_3 = (4,1)$. Now we compute the middle control points depending on the formula (2), we get

$$P_1 = (1.333333333, 3.549019607)$$
$$P_2 = (2.666666667, 3.882352941)$$

And the sum of least squares is $LSS = 0.014705882$.

Figure 2 shows the data points and fitting cubic Bezier curve with Bernstein basis and its control points.

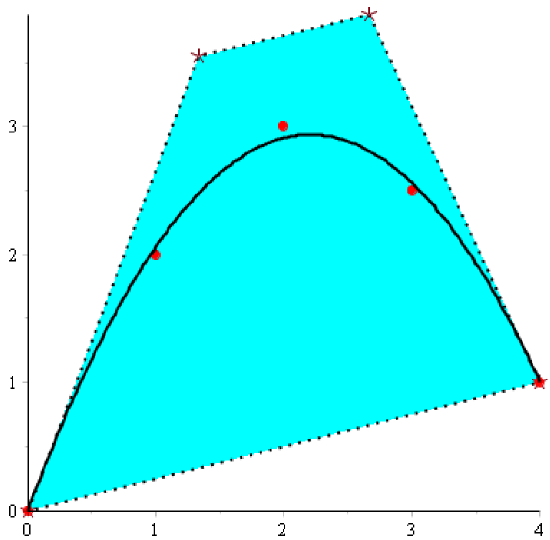

Figure 2

Now solving this problem using cubic Bezier curve with our trigonometric basis, we get

$$P_1 = (2.164784403, 4.193161191)$$
$$P_2 = (1.835215597, 4.110768980)$$

and the sum of least squares is

$$LSS = 0.014719159 .$$

Figure 3 shows the data points and fitting cubic Bezier curve with proposed trigonometric basis and its control points.

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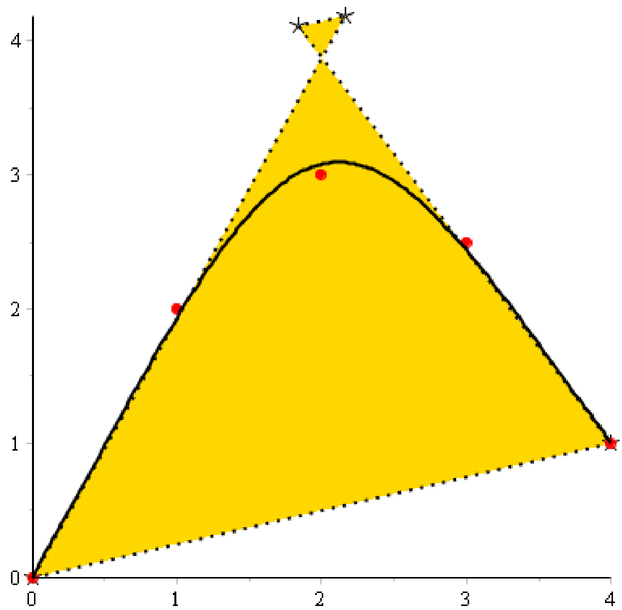

Figure 3

Example 2

Use cubic Bezier curve to fit the set of data points

$$S = \{(0,0),(0.5,1.5),(1,2.5),(2.2,3.2),(3,4),(2.5,3.5),(5,6),(5,3),(6,1.5),(6,3),(6,1)\}.$$

Solution:

Fixing first and last control points; $P_0 = (0,0)$ and $P_3 = (6,1)$.

Now using Bernstein basis, we get

$$P_1 = (1.010734406, 5.602193552)$$
$$P_2 = (6.613428009, 4.992765943)$$
$$LSS = 9.688451245$$

Figure 4 shows the data points and fitting cubic Bezier curve with Bernstein basis and its control points.

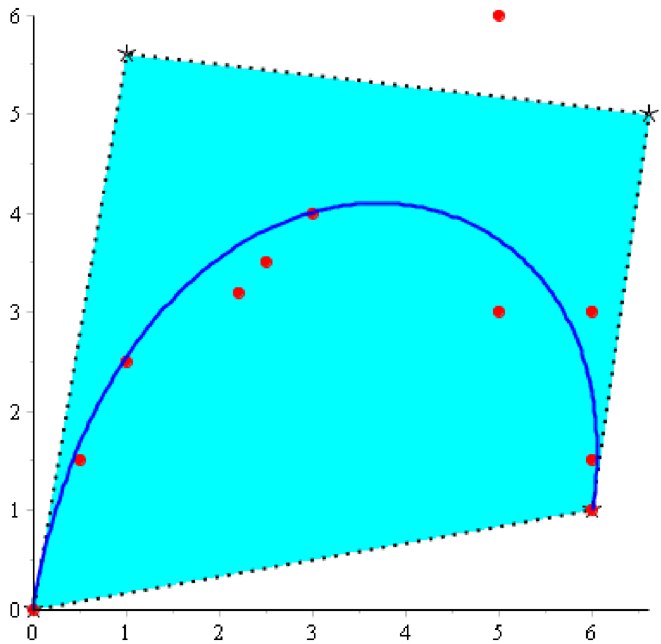

Figure 4

Now solving this problem using cubic Bezier curve with proposed trigonometric basis, we get

$$P_1 = (1.496360805, 6.706629513)$$
$$P_2 = (6.309199754, 5.247251285)$$
$$LSS = 9.510141136$$

Figure 5 shows the data points and fitting cubic Bezier curve with proposed trigonometric basis and its control points.

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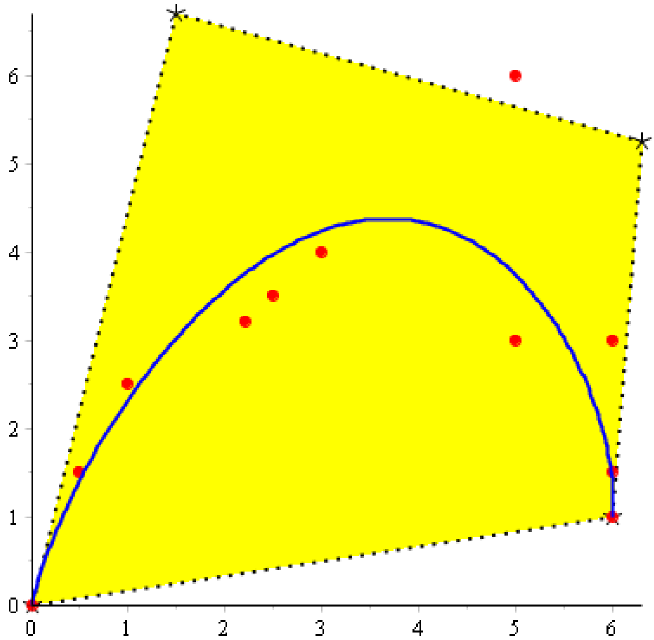

Figure 5

Example 3

Fit the set of data points using cubic Bezier curve

$S = \{(0,0), (1, 0.5), (1.2, 1.5), (1.8, 2.9), (3, 2.8), (3, 2.5), (4, 3.5), (6, 3)$

$, (7.6, 2.5), (7, 3), (8.3, 3), (10, 4)\}.$

Solution:

Using Bernstein basis, we get

$$P_1 = (1.\ 918958323, 4.950388429)$$

$$P_2 = (5.499347686, 1.273655286)$$

$$LSS = 4.635331371.$$

Figure 6 shows the data points and fitting cubic Bezier curve with Bernstein basis and its control points.

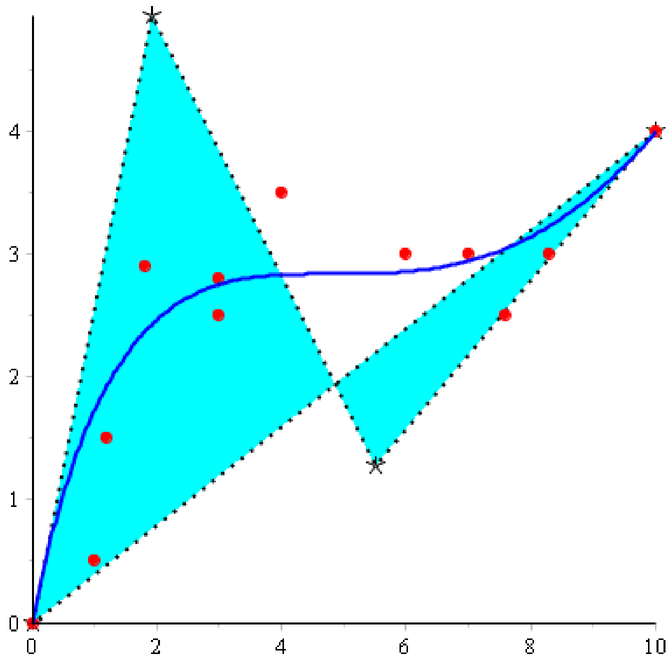

Figure 6

Now solving this problem using cubic Bezier curve with proposed trigonometric basis, we get

$$P_1 = (3.685894889, 7.290450915)$$

$$P_2 = (3.\ 372682096, -0.681272151)$$

$$LSS = 4.536009162\ .$$

Figure 7 shows the data points and fitting cubic Bezier curve with proposed trigonometric basis and its control points.

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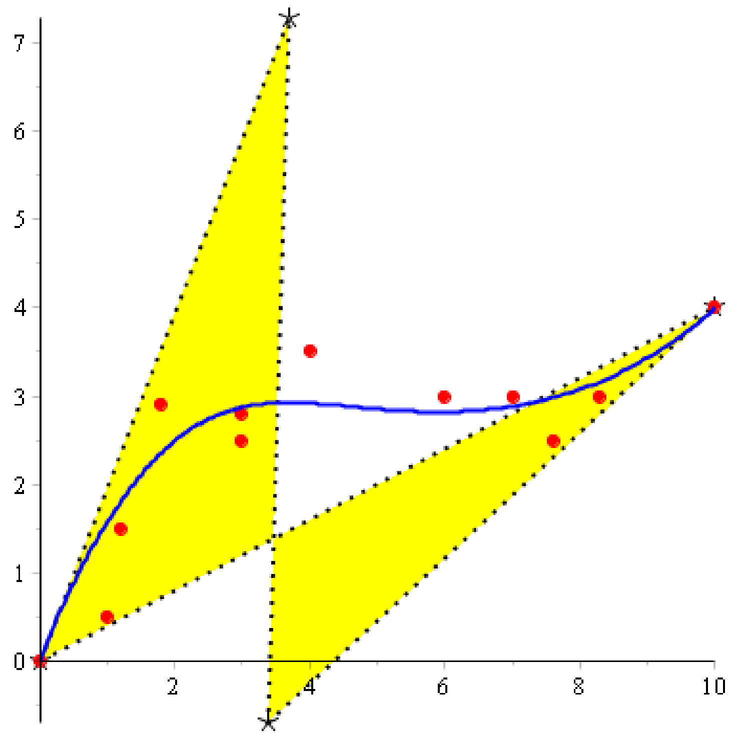

Figure 7

6–Comparison:

Considering the above examples, let us form the table

Test1: Data points: $\{(0,0),(1,2),(2,3),(3,2.5),(4,1)\}$	
Bernstein basis	
Middle control points	$P_1 = (1.333333333, 3.549019607)$ $P_2 = (2.666666667, 3.882352941)$
Least squares sum	$LSS = 0.014705882$

Trigonometric basis	
Middle control points	$P_1 = (2.164784403, 4.193161191)$ $P_2 = (1.835215597, 4.110768980)$
Least squares sum	$LSS = 0.014719159$
Test2: Data points: $\{(0,0), (0.5, 1.5), (1, 2.5), (2.2, 3.2), (3, 4), (2.5, 3.5), (5, 6), (5, 3), (6, 1.5), (6, 3), (6, 1)\}$	
Bernstein basis	
Middle control points	$P_1 = (1.010734406, 5.602193552)$ $P_2 = (6.613428009, 4.992765943)$
Least squares sum	$LSS = 9.688451245$
Trigonometric basis	
Middle control points	$P_1 = (1.496360805, 6.706629513)$ $P_2 = (6.309199754, 5.247251285)$
Least squares sum	$LSS = 9.510141136$
Test3: Data points: $\{(0,0), (1, 0.5), (1.2, 1.5), (1.8, 2.9), (3, 2.8), (3, 2.5), (4, 3.5), (6, 3), (7.6, 2.5), (7, 3), (8.3, 3), (10, 4)\}.$	
Bernstein basis	
Middle control points	$P_1 = (1.918958323, 4.950388429)$ $P_2 = (5.499347686, 1.273655286)$

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Least squares sum	$\mathrm{LSS} = 4.635331371$
Trigonometric basis	
Middle control points	$\mathrm{P}_1 = (3.685894889, 7.290450915)$ $\mathrm{P}_2 = (3.\ 372682096, -0.681272151)$
Least squares sum	$\mathrm{LSS} = 4.536009162$

The first test shows that the least squares sum in both cases are nearly identical. In the second test, the least squares sum obtained using the proposed trigonometric basis is less than that obtained using the Bernstein basis; the same observation holds for the third test.

7- Application

In digital font representation, character outlines are commonly modeled as parametric curves to achieve resolution-independent rendering and geometric precision. Among parametric representations, the cubic Bezier curve is particularly suitable due to its smoothness, local control properties, and capability to approximate complex shapes with a limited number of control points.

We will approximate the contour of the Arabic letter (Raa) by a set of cubic Bezier curve segments. Let the extracted boundary of the letter be represented as an ordered set of contour points

$$q_i = (x_i, y_i), i = 1, 2, \dots, n.$$

To perform the approximation, the contour is divided into appropriate segments, and each segment is first approximated using a cubic Bezier curve with the Bernstein basis, and then refined using the proposed trigonometric basis. [33]

Figure 6 illustrates the Arabic letter (Raa) that will be approximated using cubic Bezier curves.

Figure 6

Figure 7 shows the fitted letter (Raa) using cubic Bezier curves with Bernstein basis

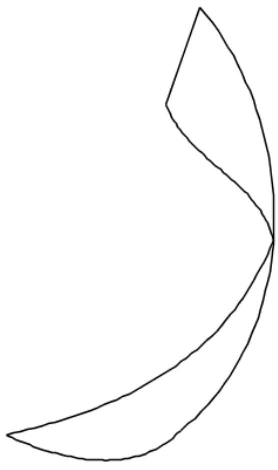

Figure 7

While Figure 8 shows the fitted letter (Raa) using cubic Bezier curves with proposed trigonometric basis

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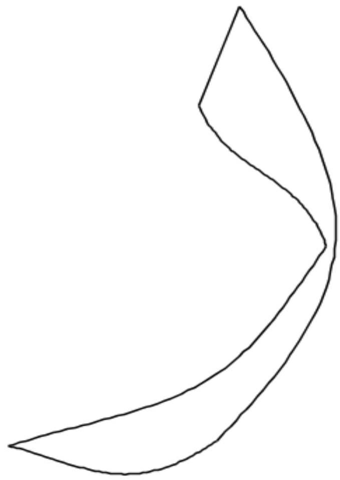

Figure 8

8- Results:

- **a)** A new parameter-free trigonometric Bernstein-like basis was successfully constructed for cubic Bezier curves, depending only on the curve parameter and degree, without introducing any additional shape parameters. This preserves simplicity and predictability while extending expressive power.
- **b)** All fundamental geometric properties of the classical Bernstein basis were rigorously preserved, including non-negativity, partition of unity, symmetry, endpoint interpolation, endpoints tangency, and smoothness, ensuring stable and geometrically meaningful curve representations.
- **c)** The proposed trigonometric basis functions were shown to belong to $C^{\infty}[0,1]$, guaranteeing infinite smoothness of the resulting Bezier-like curves and making them suitable for high-quality geometric modeling applications.
- **d)** Endpoint tangency properties were explicitly derived, demonstrating that the curve tangents at the endpoints are directly

controlled by adjacent control points, similar to classical cubic Bezier curves, thus maintaining intuitive shape control.

e) Least-squares curve fitting formulations using the proposed basis were successfully developed, leading to linear systems comparable in structure and computational cost to those of classical Bernstein-based fitting.

f) Numerical examples demonstrated that the proposed trigonometric basis consistently produces equal or smaller least-squares errors compared to the classical Bernstein basis when using the same number of control points.

g) In datasets exhibiting smooth curvature or arc-like behavior, the trigonometric basis achieved noticeably improved approximation quality, highlighting its suitability for geometries with inherent trigonometric characteristics.

h) Comparative tests confirmed that the proposed basis does not degrade performance in simple cases, as least-squares errors were nearly identical to those of the Bernstein basis when fitting simple or near-polynomial data.

i) The control polygons associated with the trigonometric Bezier curves retained strong geometric interpretability, with the curve remaining within the convex hull of its control points, ensuring predictable shape behavior.

9- Conclusion:

In this paper we introduced a new Bernstein-like trigonometric blending basis without shape parameters for constructing cubic Bezier curves. The proposed basis depends only on the curve parameter and degree, preserving the simplicity and key geometric properties of the classical Bernstein basis, including non-negativity, partition of unity, endpoint interpolation, and endpoints tangency, while incorporating the flexibility of trigonometric functions.

A theoretical and numerical comparison with the classical Bernstein basis shows that the proposed trigonometric basis provides improved approximation performance. Least-squares fitting examples demonstrate

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smaller approximation errors for the same number of control points, particularly for curves with smooth or arc-like geometry. These results suggest that the proposed parameter-free trigonometric basis is an effective alternative to classical Bernstein polynomials for curve modeling applications in computer-aided geometric design and related areas.

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